§1 Torsors [BLR 6.4]

General concept: G/S group scheme, $X, Y/S$ schemes

$G \circlearrowright X$, $X \to Y$ G-invariant.

<u>Def</u> $X \to Y$ <u>G-torsor</u> $\overset{=}{\text{def}}$

1) $G \times_S X \xrightarrow{\cong} X \times_Y X$

2) $\exists$ <u>covering</u> $Y' \to Y$ + section $Y' \times_Y X$

$$\begin{tikzcd} Y' \times_Y X \arrow[d] \\ Y' \arrow[u, bend right] \end{tikzcd}$$

↳ Usually fppf.

<u>Idea</u> Given $T \xrightarrow{y} Y$, (1) says that any two lifts $T \xrightarrow{x_1, x_2} X$

differ by <u>unique</u> $g \in G(T)$ i.e. $x_2 = g x_1$.

In other words: $\{ x \in X(T) \text{ lifting } y \}$

is $\emptyset$ or $\cong G(T)$

(2) says: $\exists$ covering $T' \to T$ + $x \in X(T')$ lifting y.

(Excludes e.g. situation $X = \emptyset$.)

"Covering" has to be specified, usually fppf.

Example 1) $X = \operatorname{Spec} L \longrightarrow Y = \operatorname{Spec} K$.

finite Galois w/ Gal. grp. $\Gamma \circlearrowright X$

↳ torsor for étale topology:

$$L \otimes_K X = \operatorname{Spec}(L \otimes_K L) = \underset{\circlearrowright}{\Gamma} \times \operatorname{Spec} L$$

Γ acts by translation.

2) $X \longrightarrow Y / \operatorname{Spec} k$ isogeny of ab vars w/ kernel K.

$$K \times_{\operatorname{Spec} k} X \xrightarrow{\cong} X \times_Y X.$$

So 1) satisfied. 2) satisfied w/ fppf topology & $Y' = X$ itself.

3) Thm (AV Lect 14) G/S fin loc free, X/S sep.
$G \circlearrowright X$ freely (i.e. $G \times_S X \hookrightarrow X \times_S X$ cl. immers.)
Assume $\exists$ affine G-stable cover of X

Then Quotient Y exists, $X \to Y$ is fin loc free

and $G \times_S X \xrightarrow{\cong} X \times_Y X$. In other words,
$X \to Y$ is G-torsor for fppf topology.

§2 $\mathbb{G}_m$-torsors

Fact: $X \to Y$ GL_n-torsor for ét / fppf / fpqc top, then $\exists$ Zariski covering $Y = \cup U_i$ s.th. $X(U_i) \neq \emptyset$.

(This is a consequence of fpqc descent for vector bundles. It means torsors for different topologies coincide.)

$$\{\mathbb{G}_m\text{-torsors } X \xrightarrow{\pi} Y\} \xleftarrow{\cong} \mathrm{Pic}(Y)$$

$$\underline{\mathrm{Spec}} \bigoplus_{i \in \mathbb{Z}} \mathcal{L}^{\otimes i} \longleftarrow\!\shortmid \mathcal{L}$$

$$X \longmapsto \{ l \in \pi_* \mathcal{O}_X \mid \mu^*(l) = t \otimes l \}.$$

Setting $X = \mathrm{Spec}\, A$, (over $S = \mathrm{Spec}\, R$) $\mathbb{G}_m \circlearrowright X$ $\longleftrightarrow$ $A = \bigoplus_{i \in \mathbb{Z}} A_i$

Action free $\overset{=}{\text{def}}$ $\mathbb{G}_m \times X \to X \times X$ closed immersion.

Prop If action free, quotient $q : X \to Y = \mathrm{Spec}\, A_0$ is a $\mathbb{G}_m$-torsor. More precisely, A_1 is a line bundle $/A_0$ and $A = \bigoplus_{i \in \mathbb{Z}} A_1^{\otimes i}$ as rings.

Proof $\mathbb{Z}[t^{\pm 1}] \underset{\mathbb{Z}}{\otimes} A \longleftarrow A \underset{\mathbb{Z}}{\otimes} A$

$t^d \otimes a_i b \longleftarrow a \otimes b \qquad \deg(a) = d$

being surjective $\Rightarrow$ $t \otimes 1$ in image, i.e.

$\exists\ e_1, \dots, e_r \in A_1,\ f_1, \dots, f_r \in A_{-1}$

s.th. $1 = \sum e_i f_i \qquad u_i = e_i f_i$

Note $\deg(u_i) = 0$, so $\operatorname{Spec} A_0 = \bigcup D(u_i)$

Claim $D(u_i) \underset{Y}{\times} X \overset{?}{\cong} \mathbb{G}_{m, D(e_i f_i)}$

$\mathbb{G}_m$-equivariantly.

Proof u_i invertible $\Rightarrow$ e_i, f_i invertible

Given $a \in A_1[u_i^{-1}]$ may write

$$a = a\left(\sum e_j f_j\right) = \underbrace{a \cdot \left(\sum \frac{e_j f_j}{e_i}\right)}_{\in A_0} \cdot e_i$$

So $A_1[u_i^{-1}] = A_0[u_i^{-1}] e_i$ since e_i invertible in $A[u_i^{-1}]$,

even $A_0[u_i^{-1}] \xrightarrow[\cong]{\cdot e_i} A_1[u_i^{-1}]$

$\Longrightarrow$ A_1 l.b. over A_0, trivialized by e_i over $D(u_i)$.

Given $a \in A_d[u_i^{-1}]$, $\quad a = (a \cdot e_i^{-d}) \cdot e_i^d$
$\quad \in A_0[u_i^{-1}] \cdot e_i^d$,

Thus the natural map

$$\bigoplus_{i \in \mathbb{Z}} A_1^{\otimes i} \longrightarrow A \quad \text{is an iso.} \qquad \square$$

Important consequence:

$X \to Y$ has Zariski locally sections & these are unique up to $\mathbb{G}_m$-action.

<u>Example</u> $\widetilde{\mathcal{M}} \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}} = \operatorname{Spec} \mathbb{Z}[\frac{1}{6}, j]$

does not have sections everywhere

Recall first that if K field, $j \in K$, then $\exists$ EC E/K with $j(E) = j$.

$\Rightarrow \exists\ E/\mathbb{Q}(j)$ with $j(E) = j$.

It extends to an open $U \subseteq \mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}$

<u>Claim</u> $\nexists$ EC E/R $\quad R = \mathbb{Q}[j]_{(j)}$ s.t. $j(E) = j$.

(R = local ring at $j=0$ in $\mathbb{A}^1_{\mathbb{Q}}$)

Proof R local, so E would be given by Weierstraß equn

$$y^2 = x^3 + ax + b \quad , \quad a, b \in R.$$

Then $\Delta = (\text{const} \neq 0) + \text{higher in } \hat{R} = \mathbb{Q}[[j]]$

as it lies in $\hat{R}^\times$.

But $j = 1728 \frac{4a^3}{\Delta}$ has no solution in $\hat{R}$. $\square$

Rmk $j = 0 \longleftrightarrow y^2 = x^3 + 1$

has extra auto $\begin{array}{l} x \mapsto \zeta_3 x \\ y \mapsto y \end{array} \Big/ \mathbb{Q}(\zeta_3)$

CM by $\mathbb{Z}[\zeta_3]$

Similar argument works at

$j = 1728 \longleftrightarrow y^2 = x^3 + x$

has auto $\begin{array}{l} x \mapsto -x \\ y \mapsto iy \end{array} \Big/ \mathbb{Q}(i)$

CM by $\mathbb{Z}[i]$.

§3 Level structure $\quad n \geq 1, \quad n \in \mathcal{O}_S(S)^\times$.

$E \to S$ EC. Then $E[n] \to S$ fin. ét. order n^2.

Two important principles:

1) $X \xrightarrow{f} Y$ fin ét. Then f open + closed

.) closed since finite

.) open since flat, loc fin. pres. maps are open
(Stacks 01UA.)

2)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u"'] & & Y \arrow[dl, "v"] \\ & S & \end{tikzcd}$$
u, v fin ét $\Longrightarrow$ f fin étale.

Example $\quad X \to S$ fin. ét. of deg d.

Then $\exists$ $S' \to S$ fin ét s.th. $S' \times_S X \cong \coprod S'$
$$\begin{tikzcd} S' \times_S X \arrow[rr, "\cong"] \arrow[dr] & & \coprod S' \arrow[dl] \\ & S' & \end{tikzcd}$$

Proof by induction on d, $d=1$ being $X = S$.

$\Delta : X \to X \times_S X$ fin. ét. by principle 2),

hence $X \times_S X = \Delta \amalg (\text{rest})$ by principle 1). $\square$

Prop $n \in \mathcal{O}_S(S)^\times$, E/S EC. Then $\exists\, S' \to S$ fin ét. s.th. $S' \times_S E[n] \cong \underline{(\mathbb{Z}/n)^{\oplus 2}}_{S'}$.

Proof of Prop Example yields $S' \to S$ fin. ét.

$$S' \underset{S}{\times} E[n] \cong \coprod_{i \in I} S' \qquad |I| = n^2$$

Denote sections by t_i, $i \in I$.

For $a,b \in \mathbb{Z}/n$, $S'_{a,b,i,j} = (a t_i + b t_j) \cap e(S') \subseteq S'$ is open + closed, since $(a t_i + b t_j) \cap e(S') \to e(S')$ is fin. étale.

$$\Rightarrow S' = S'_{a,b,i,j} \amalg S'^{\,c}_{a,b,i,j}$$

Now given $s \in S'$, we know $\operatorname{Spec} \mathcal{K}(s) \underset{S'}{\times} E[n] = (\mathbb{Z}/n)^{\oplus 2}$.

Pick i,j s.th. t_i, t_j provide basis in s.

Only fin many a,b exist

$$\Rightarrow \exists\; S' = S'_{i,j} \amalg S'^{\,c}_{i,j} \text{ where}$$

$$t_i, t_j : \underline{\mathbb{Z}/n}^{\oplus 2}_{S'_{i,j}} \xrightarrow{\cong} S'_{i,j} \underset{S'}{\times} E[n]$$

Varying s finishes the proof. $\square$

<u>Prop</u> E/S as before. The functor

$$L_{E,n} : Sch/S^{op} \longrightarrow Set$$
$$T/S \longmapsto \left\{ \alpha : \underline{(\mathbb{Z}/n)}^{\oplus 2}_T \xrightarrow{\cong} T \times_S E[n] \right\}$$

is representable + fin. étale $/S$. $L_{E,n} \to S$ is

a $GL_2(\mathbb{Z}/n)$-torsor (for étale topology).

<u>Rmk</u> 1) Group homo $\alpha : \underline{\mathbb{Z}/n}^{\oplus 2}_T \longrightarrow T \times_S E[n]$ is

same as two $\alpha_1, \alpha_2 \in E[n](T)$.

2) $GL_2(\mathbb{Z}/n) \subset L_{E,n}$ as $g \cdot \alpha = \alpha \circ g$.

Proof Consider $X := E[n] \times_S E[n]$.

Then $X(T) = \mathrm{Hom}\left(\underline{\mathbb{Z}/n}_T^{\oplus 2}, T \times_S E[n] \right)$

$\forall a,b \in (\mathbb{Z}/n)^2 \setminus \{(0,0)\}$, have

$$m_{a,b} : X \longrightarrow E[n]$$
$$(\alpha_1, \alpha_2) \longmapsto (a\alpha_1, b\alpha_2)$$

It is finite étale. Then

$$B_{a,b} := X \underset{m_{a,b}, E[n], e}{\times} S \longrightarrow X$$

is open and closed.

$B_{a,b}$ = locus where a,b give non-trivial dependence relation of α_1, α_2.

$$X \setminus \bigcup_{(a,b) \neq (0,0)} B_{a,b} = L_{E,n} \text{ is}$$

sought for space.

We know $\exists\, T \to S$ fin étale s.th.

$$T \times_S E[n] \cong \underline{\mathbb{Z}/n}_T^{\oplus 2}.$$

So $T \times_S \underline{\mathrm{Iso}}\left(\underline{\mathbb{Z}/n}_S^{\oplus 2}, E[n] \right)$ giving torsor property.

$\cong \underline{\mathrm{Iso}}\left(\underline{\mathbb{Z}/n}_T^{\oplus 2}, \underline{\mathbb{Z}/n}_T^{\oplus 2} \right) \cong \underline{GL_2(\mathbb{Z}/n)}_T$, $\square$

Def Iso of ECs w/ level-n-str $(E,\alpha), (E',\alpha')$

$\underset{\text{def}}{=}$ Iso $\phi : E_1 \xrightarrow{\cong} E_2$ s.th. $\alpha_2 = \phi \circ \alpha_1$

$$\mathcal{M}_n : \mathrm{Sch}/_{\mathbb{Z}[\frac{1}{n}]}^{op} \longrightarrow \mathrm{Set}$$
$$S \longmapsto \{(E,\alpha)/S\}/\cong$$

$$\widetilde{\mathcal{M}}_n : \mathrm{Sch}/_{\mathbb{Z}[\frac{1}{6n}]}^{op} \longrightarrow \mathrm{Set}$$
$$S \longmapsto \left\{ (E,\alpha,\pi) \;\middle|\; \begin{array}{l} (E,\pi) \in \widetilde{\mathcal{M}}(S), \\ \alpha \in L_{E,n}(S) \end{array} \right\} / \cong$$

<u>Cor</u> $\widetilde{\mathcal{M}}_n$ is an affine scheme

<u>Proof</u> $\widetilde{\mathcal{M}}_n = L_{\mathcal{E},n} \longrightarrow \widetilde{\mathcal{M}}[\frac{1}{n}]$, $(\mathcal{E},\pi)/\widetilde{\mathcal{M}}$ universal curve

is representable. (Even $GL_2(\mathbb{Z}/n)$-torsor.) $\square$

<u>Rmk</u> Def makes sense in char p, $p \mid n$, too,

but produces <u>empty functor</u> since these

$E[n]$ is never étale.

<u>Obtain</u>

$$\begin{tikzcd} & \widetilde{\mathcal{M}}_n \arrow[dl] \arrow[dr, "\text{forget } \pi"] & \\ \widetilde{\mathcal{M}}_n / \mathrm{Chm} & & \mathcal{M}_n[\tfrac{1}{6}] \end{tikzcd}$$

<u>Thm</u> Assume $n \geq 3$. Then $\widetilde{\mathcal{M}}_n / \mathrm{Chm} \cong \mathcal{M}_n[\frac{1}{6}]$.

In ptic, $\mathcal{M}_n$ is (representable by an) affine scheme.

Following prop explains why thus is plausible:

<u>Prop</u> $n \geq 3$, $(E, \alpha)/S$ EC + level-n-str.

Then $\mathrm{Aut}(E, \alpha) = \{ \mathrm{id} \}$.

<u>1st Proof</u> First observe: If S connected, $\phi : E \to E'$ map of ECs over S s.th. $\phi(s) = 0$ for some $s \in S$, then $\phi = 0$ by Rigidity. $\Rightarrow$ wlog $S = \operatorname{Spec} k$ char $k \nmid n$.

To see: $n \geq 3 \Rightarrow \mathrm{End}(E) \xrightarrow{r} \mathrm{End}(E[n](\bar{k}))$ is injective on $\mathrm{Aut}(E)$.

Let $\phi \in \mathrm{Aut}(E)$. Classif. of $\mathrm{End}(E)$

$\Rightarrow \mathbb{Z}[\phi] \in \{ \mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\zeta_3] \}$

with $\phi \in \{1, -1\}$ or $\{\pm i\}$ or $\{\zeta_3^{\pm 1}, \zeta_6^{\pm 1}\}$

If $\phi = -1$, then $\tau(\phi) = -1$, which is $\not\equiv 1$ mod n, so we are good in case $\phi \in \mathbb{Z}$.

Assume $\mathbb{Z}[\phi] =: R$ is quadratic.

<u>Claim</u> $E[n](k) \cong R/nR$ as R-module.

<u>Proof</u> Seen last term:

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell E)$$

Our R is also the <u>max order</u>, i.e. normal, so $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R = \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ or DVR.

Any torsion-free finite module over DVR is free, so $T_\ell E$ is free (necessarily rank 1) over $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R$.

Truncating, we find $E[\ell^e](k) \cong R/\ell^e$.

CRT $\Rightarrow$ $\square$ Claim.

Now $1 \not\equiv \pm i \mod n \cdot \mathbb{Z}[i]$ because

$(n = 2^e)$ $\quad \mathbb{Z}[i]$ ramified at 2, $1 \pm i$ are uniformizers

$(2 \neq p \mid n)$ $\quad$ primitive k-th roots of 1, $p \nmid k$, different in $\overline{\mathbb{F}_p}$

Similarly $1 \not\equiv \zeta_3^{\pm 1}, \zeta_6^{\pm 1} \mod n\,\mathbb{Z}[\zeta_3]$ because

$(n = 3^e)$ $\quad \mathbb{Z}[\zeta_3]$ ramified at 3, $1 - \zeta_3^{\pm 1}$, $\zeta_6^{\pm 1} - 1$ uniformizers

$(3 \neq p \mid n)$ $\quad$ same argument. $\square$

<u>2nd Proof</u> Assume $\phi|_{E[n]} = 1$, i.e. $E[n] \subset \ker(\phi - 1)$.

Consider

$$\begin{tikzcd} 0 \arrow[r] & E[n] \arrow[r] & E \arrow[r, "\cdot n"] \arrow[d, "\phi - 1"'] & E \arrow[r] \arrow[dl, dashed, "\exists \psi"] & 0 \\ & & E & & \end{tikzcd}$$

$E \,/\!/\, E/E[n]$ as seen last term

$\exists \psi$ by quotient property.

i.e. $(\phi - 1) = n \cdot \psi$. Then

$$n^2 \deg(\psi) = (\phi - 1)(\phi^* - 1) = \deg \phi - (\phi + \phi^*) + 1$$

$$= 2 - (\phi + \phi^*).$$

Classification of End(E) $\Rightarrow$ $|\phi + \phi^*| \leq 2$

So $n^2 \deg \psi \leq 4$.

This forces $\psi = 0$ if $n \geq 3$. $\square$

§4 Proof of Thm $n \geq 3$, over $\mathbb{Z}[\frac{1}{6n}]$

Observe If $\mathcal{M}_n[\frac{1}{6}]$ representable, then

$\widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ is $\mathbb{G}_m$-torsor since

1) $\mathbb{G}_m \times \widetilde{\mathcal{M}}_n \xrightarrow{\cong} \widetilde{\mathcal{M}}_n \times_{\mathcal{M}_n} \widetilde{\mathcal{M}}_n$

2) $\widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ has Zariski loc sections
(trivialize $\omega_{\mathcal{E}}$,
$(\mathcal{E}, \alpha)$ universal curve)

$\Rightarrow$ Necessarily $\mathcal{M}_n = \mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n$!

Assume we know $\mathbb{G}_m \subset \widetilde{\mathcal{M}}_n$ freely. Then

$q : \widetilde{\mathcal{M}}_n \to \mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n$ is $\mathbb{G}_m$-torsor (§2),

hence has local sections, unique up to $\mathbb{G}_m$.

$$\mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathcal{M}_n$$

1) $y \in (\mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n)(S)$ $\Phi(y)$: Pick $S = S_i$
+ (E_i, α_i, π_i) lifting $y|_{S_i}$

Torsor-property $\Rightarrow$ $(E_i, \alpha_i, \pi_i)|_{S_{ij}} = (E_j, \alpha_j, \lambda_{ij}\pi_j)|_{S_{ij}}$

for unique $\lambda_{ij} \in \mathcal{O}_S(S_{ij})^\times$.

Translation: $\exists! \ \phi_{ij} : (E_i, \alpha_i)|_{S_{ij}} \xrightarrow{\cong} (E_j, \alpha_j)|_{S_{ij}}$

Cocycle condition satisfied since $\mathrm{Aut}(E, \alpha) = \{id\}$
(use $n \geq 3$)

$\Rightarrow$ Glue to $\Phi(y) := (E, \alpha) \in \mathcal{M}_n(S)$.

2) $(E, \alpha) \in \mathcal{M}_n(S)$ $\Psi(E, \alpha)$: Pick $S = S_i$ s.t.

$\omega_E|_{S_i} \cong \mathcal{O}_{S_i}$, lifts $(E, \alpha, \pi_i) \in \widetilde{\mathcal{M}}_n(S_i)$.

Since q $\mathbb{G}_m$-invariant, $\{q(E, \alpha, \pi_i)\}_i$
agree on overlaps, define $S \longrightarrow \mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n$.

3) $\Psi \circ \Phi = id_{\mathbb{G}_m \backslash \widetilde{\mathcal{M}}_n}$, $\Phi \circ \Psi = id_{\mathcal{M}_n}$

Given y & lifts (E_i, α_i, π_i) gluing to (E, α),
the (E_i, α_i, π_i) lift (E, α) and glue to y.
Other identity similar. $\square$

§5 Weil extension Thm

<u>To see</u>: $\mathbb{G}_m \times \widetilde{\mathcal{M}}_n \longrightarrow \widetilde{\mathcal{M}}_n \times_{\mathbb{Z}[\frac{1}{6n}]} \widetilde{\mathcal{M}}_n$ is

closed immersion, i.e. proper monomorphism

[Stacks 04XV]

<u>Monomorphism</u> $\lambda, \lambda' \in \mathbb{G}_m(S)$, $(E, \alpha, \pi), (E', \alpha', \pi') \in \widetilde{\mathcal{M}}_n(S)$.

Then $(E, \alpha, \pi) \underset{\exists \phi}{\cong} (E', \alpha', \pi')$ & $(E, \alpha, \lambda\pi) \underset{\exists \psi}{\cong} (E', \alpha', \lambda'\pi')$ $\Big\} \Rightarrow \lambda = \lambda'$

<u>Proof</u> $n \geq 3 \Rightarrow \phi = \psi$ since $\exists_{\leq 1}$ iso $(E, \alpha) \xrightarrow{\cong} (E', \alpha')$.

So $\phi^*(\pi') = \pi$ & $\lambda' \phi^*(\pi') = \lambda \pi$

$\Rightarrow \lambda = \lambda'$ $\square$

<u>Properness</u> <u>To show</u> R DVR, $K = \operatorname{Frac} R$.

$(E, \alpha, \pi), (E', \alpha', \pi') \in \widetilde{\mathcal{M}}_n(R)$

s.th. $\exists$ iso $\phi_K : (E, \alpha)_K \xrightarrow{\cong} (E', \alpha')_K$ with $\phi^*(\pi') = \lambda \cdot \pi$ $\lambda \in K^\times$

Then ϕ_K lifts uniquely to $(E,\alpha) \to (E',\alpha')$ & $\lambda \in R^\times$.

<u>Thm (Weil extension Thm)</u> S Dedekind scheme, connected, η gen pt.

$E, E'/S$ ECs. Then

$$\mathrm{Hom}(E,E') \xrightarrow{\cong} \mathrm{Hom}(E_\eta, E'_\eta)$$

<u>Rmk</u> E_η, E'_η are schematically dense & E' separated, so <u>injectivity</u> is immediate. <u>Surjectivity</u> is the real statement here.

<u>Implies desired properness</u> $\phi: E \to E'$ the unique lift of ϕ_K. (Is iso since also ϕ_K^{-1} lifts.)

Then $\phi \circ \alpha = \alpha'$ since $E[n]_K, E'[n]_K, (\underline{\mathbb{Z}/n})^{\oplus 2}_K$ are schematically dense and $\phi_K \circ \alpha_K = \alpha'_K$.

Also $\phi^*(\pi') = \mu \cdot \pi$ for some $\mu \in R^\times$.

Then $\mu = \lambda$ since $\Gamma(E, \Omega^1_{E/R}) \hookrightarrow \Gamma(E_K, \Omega^1_{E/K})$.